Polyhedral Volumes

Visual Techniques

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Outline

- Identities of the golden ratio.

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- Locating coordinates of regular polyhedra.

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- Volume of the icosahedron.

The Golden Ratio

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- The golden ratio and its *scaling* property.
- The *scaling* rule for areas and volumes.
- The Pythogorian theorem.
- Formula for pyramid volume.

Basic Units

Color	Significance	1	2
Blue	Unity	1	ϕ
Red	Radius of I_1	$\sin 72$	$\phi \sin 72$
Yellow	Radius of C_1	$\sin 60$	$\phi \sin 60$
Green	Face diagonal of C_1	$\sqrt{2}$	$\phi\sqrt{2}$

Powers Of The Golden Ratio

$$
\begin{aligned}
1 + \phi &= \phi^2 \\
\phi + \phi^2 &= \phi^3 \\
\vdots &= \vdots \\
\phi^{n-2} + \phi^{n-1} &= \phi^n
\end{aligned}
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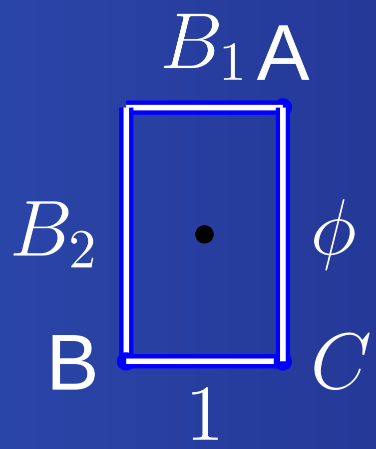
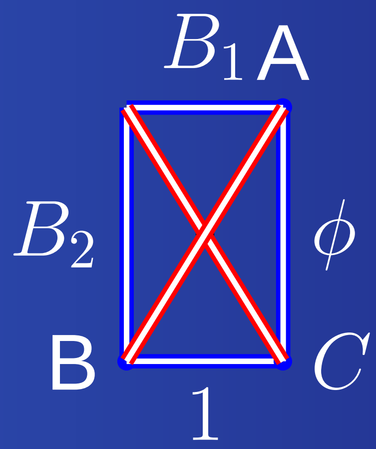
Form a Fibonacci sequence.

Golden Rhombus



Golden Rhombus



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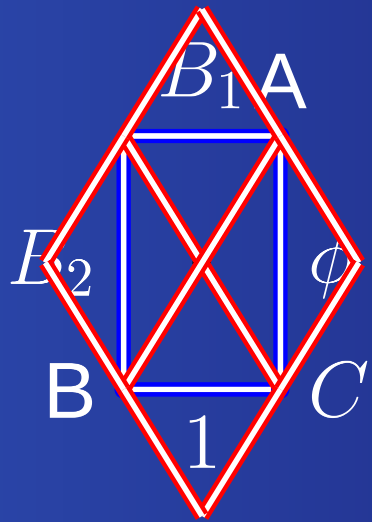
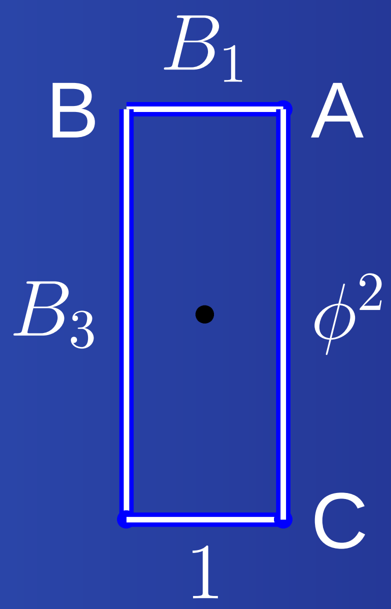


Scaled Golden Rhombus



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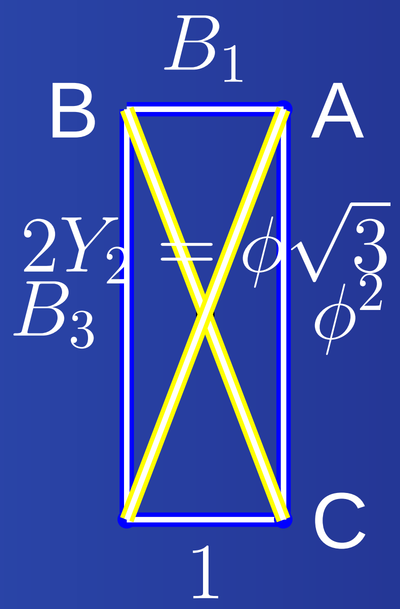
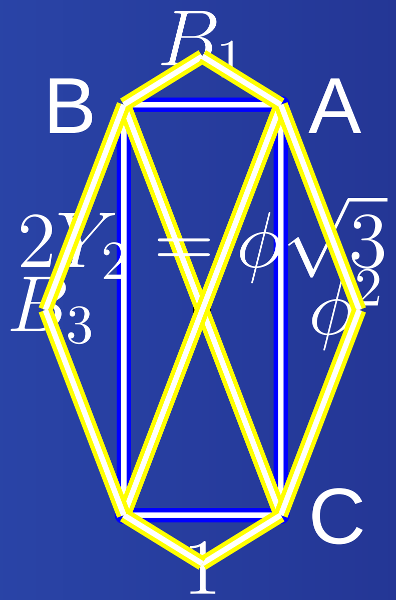


Scaled Golden Rhombus



Useful Identities

$$2\cos 36 = \phi$$

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Golden ratio and pentagon diagonal.

Useful Identities

$$1 + \phi^2 = 4{R_1}^2$$

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Blue-red triangle.

Useful Identities

$$\cos 2 * 18 = 2\cos^2 18 - 1$$
$$= 2\sin^2 72 - 1$$

Combining these gives

$$\sin^2 72 = \frac{1+\phi^2}{4}$$
$$= R_1^2$$

Useful Identities

$$1 + \phi^4 = 3\phi^2$$

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Blue-yellow triangle.

Useful Identities

$$\sin 36 = \frac{\sqrt{1+\phi^2}}{2\phi}$$

Locating Vertices Of Regular Polyhedra

Cube

$$\{(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})\}.$$

Tetrahedron

$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$
$(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$	$(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$

Self dual.

Octahedron

$$\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}.$$

Dual To Cube

Rhombic Dodecahedron

$$(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}).$$

Vertices of cube and octahedron.

$$\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}.$$

Cube-Octahedron

$$\{(0, \pm 1, \pm 1), (\pm 1, 0, \pm 1), (\pm 1, \pm 1, 0)\}.$$

Dual to rhombic dodecahedron.

Faces of cube and octahedron.

Dodecahedron

Cube vertices

$$(\pm\frac{\phi}{2}, \pm\frac{\phi}{2}, \pm\frac{\phi}{2})$$

Coordinate planes.

$$(\pm\frac{\phi^2}{2}, \pm\frac{1}{2}, 0) \quad (\pm\frac{1}{2}, 0, \pm\frac{\phi^2}{2}) \quad (0, \pm\frac{\phi^2}{2}, \pm\frac{1}{2})$$

Icosahedron

Dual to dodecahedron.

$$\left\| \begin{array}{l} (\pm\frac{\phi}{2}, \pm\frac{1}{2}, 0) \\ (0, \pm\frac{\phi}{2}, \pm\frac{1}{2}) \\ (\frac{1}{2}, 0, \pm\frac{\phi}{2}) \end{array} \right\|$$

Using The Cube To Compute Volumes

Volume Of The Tetrahedron

Constructing right-angle pyramids on tetrahedral faces forms a cube.

$$\frac{1}{2}\frac{1}{3} = \frac{1}{6}.$$

$$\begin{aligned} V_T &= 1^3 - \frac{4}{6} \\ &= \frac{1}{3}. \end{aligned}$$

Volume Of The Octahedron

Place 4 tetrahedra on 4 octahedral faces to form a $2x$ tetrahedron.

Octahedron is 4 times the tetrahedron.

$$\begin{aligned} V_O &= \frac{8}{3} - 4\frac{1}{3} \\ &= \frac{4}{3}. \end{aligned}$$

Volume Of The Rhombic Dodecahedro

- Connect the center of the cube to its vertices.
- This forms 6 pyramids inside the cube.

Volume Of The Cube-octahedron

Subtracting 8 right-angle pyramids from a cube gives a cube-octahedron.

$$V_{\text{CO}} = 8 - 8\frac{1}{6}$$
$$= \frac{20}{3}.$$

Volume Of The Dodecahedron

Cube And The Dodecahedron

Dodecahedron contains a golden cube.

- 8 of the 20 vertices determine a cube.
- Cube edges are dodecahedron face diagonals.

Constructing Dodecahedron From A C

- Consider again the golden cube.
- Construct *roof* structures on each cube face.

Unit dodecahedron around a golden cube.

Summing The Parts

- Volume of the golden cube is ϕ^3.
- Consider the *roof* structure.

Volume Of Pyramid

- Pyramid has rectangular base.
- Rectangle of side $\phi \times \frac{1}{\phi}$.

Volume is $\frac{1}{6}$.

Triangular Cross-Section

- Cross-section has length 1.
- Triangular face with base ϕ,

Volume is $\frac{\phi}{4}$*.*

Dodecahedron Volume

$$\phi^3 + 6(\frac{\phi}{4} + \frac{1}{6})$$

Volume Of The Icosahedron

Volume Of The Icosahedron

- Icosahedron is dual to dodecahedron.
- Octahedron is dual to the cube.

Octahedron outside *icosahedron gives volume.*

Constructing The Octahedron

Squares in XY, YZ, and ZX planes.

- Consider a pair of opposite icosahedral edges,
- And construct right-triangles in their plane,

Square In XY Plane

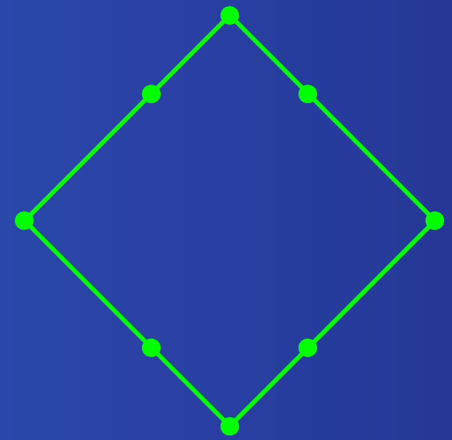

Figure 1: Green square around a blue golden rectangle.

Square In XY Plane

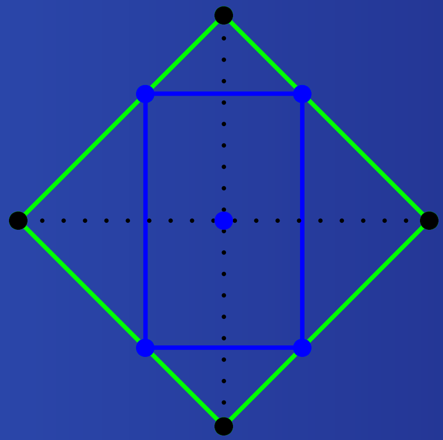

Figure 1: Green square around a blue golden rectangle.

Complete The Octahedron

- Construct similar squares in the YZ and ZX planes.
- Constructs an octahedron of side $\frac{\phi^2}{\sqrt{2}}$.

Volume is $\frac{\phi^6}{6}$.

Computing The Residue

- Icosahedron embedded in this octahedron.
- Icosahedral volume found by subtracting residue from $\frac{\phi^6}{6}$.

Pyramid Volume

- Observe pyramid with right-triangle base in XY plane.
- Triangular base has area $\frac{1}{4}$.

Pyramid Volume is $\frac{\phi}{24}$

Icosahedral Volume

$$\frac{\phi^6}{6} - \frac{\phi}{2}$$

Conclusion